On some commutator theorems for fractional integral operators on the weighted Morrey spaces

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Abstract

Let $0 < \alpha < n$ and I_α be the fractional integral operator. In this paper, we will show some weighted boundedness properties of commutator $[b, I_\alpha]$ on the weighted Morrey spaces $L^{p,\kappa}(w)$ under appropriate conditions on the weight w, where the symbol b belongs to weighted BMO or Lipschitz space or weighted Lipschitz space.

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1. Introduction

The classical Morrey spaces $\mathcal{L}^{p,\lambda}$ were originally introduced by Morrey in [7] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7,11]. In [1], Chiarenza and Frasca showed the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces.

Recently, Komori and Shirai [6] defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of the above classical operators on these spaces. Assume that I_α is a fractional integral operator and b is a locally integrable function on $\mathbb{R}^n$, the commutator of b and I_α is defined as follows

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x).$$

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In [6], the authors proved that when $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $0<\kappa<p/q$ and $w\in A_{p,q}$(Muckenhoupt weight class), then $[b,I_\alpha]$ is bounded from $L^{p,\kappa}(w^p,w^q)$ to $L^{q,\kappa q/p}(w^q)$ whenever $b\in BMO(\mathbb{R}^n)$.

The main purpose of this paper is to study the weighted boundedness of commutator $[b,I_\alpha]$ on the weighted Morrey spaces when b belongs to some other function spaces. Our main results are stated as follows.

Theorem 1. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $0<\kappa<p/q$ and $w^{q/p}\in A_1$. Suppose that $b\in BMO(w)$(weighted BMO) and $r_w>\frac{1-\kappa}{p/q-\kappa}$, then $[b,I_\alpha]$ is bounded from $L^{p,\kappa}(w)$ to $L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)$, where r_w denotes the critical index of w for the reverse Hölder condition.*

Theorem 2. *Let $0<\beta<1$, $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/s=1/p-(\alpha+\beta)/n$, $0<\kappa<\min\{p/s,p\beta/n\}$ and $w^s\in A_1$. Suppose that $b\in Lip_\beta(\mathbb{R}^n)$(Lipschitz space), then $[b,I_\alpha]$ is bounded from $L^{p,\kappa}(w^p,w^s)$ to $L^{s,\kappa s/p}(w^s)$.*

Theorem 3. *Let $0<\beta<1$, $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/s=1/p-(\alpha+\beta)/n$, $0<\kappa<p/s$ and $w^{s/p}\in A_1$. Suppose that $b\in Lip_\beta(w)$(weighted Lipschitz space) and $r_w>\frac{1}{p/s-\kappa}$, then $[b,I_\alpha]$ is bounded from $L^{p,\kappa}(w)$ to $L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)$.*

2. Definitions and Notations

First let us recall some standard definitions and notations of weight classes. A weight w is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere, all cubes are assumed to have their sides parallel to the coordinate axes. Given a cube Q and $\lambda>0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q, $Q=Q(x_0,r)$ denotes the cube centered at x_0 with side length r. For a given weight function w, we denote the Lebesgue measure of Q by $|Q|$ and the weighted measure of Q by $w(Q)$, where $w(Q)=\int_Q w(x)\,dx$.

We shall give the definitions of three weight classes as follows.

Definition 1 ([8])**.** *A weight function w is in the Muckenhoupt class A_p with $1<p<\infty$ if for every cube Q in $\mathbb{R}^n$, there exists a positive constant C which is independent of Q such that*

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}\le C.$$

When $p=1$, $w\in A_1$, if

$$\frac{1}{|Q|}\int_Q w(x)\,dx\le C\,\underset{x\in Q}{ess\,inf}\,w(x).$$

When $p=\infty$, $w\in A_\infty$, if there exist positive constants δ and C such that given a cube Q and E is a measurable subset of Q, then

$$\frac{w(E)}{w(Q)}\le C\left(\frac{|E|}{|Q|}\right)^{\delta}.$$

Definition 2 ([9])**.** *A weight function w belongs to $A_{p,q}$ for $1<p<q<\infty$ if for every cube Q in $\mathbb{R}^n$, there exists a positive constant C which is independent of Q such that*

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w(x)^{-p'}\,dx\right)^{1/p'}\le C,$$

where p' denotes the conjugate exponent of $p>1$; that is, $1/p+1/p'=1$.

Definition 3 ([3])**.** *A weight function w belongs to the reverse Hölder class RH_r if there exist two constants $r>1$ and $C>0$ such that the following reverse Hölder inequality*

$$\left(\frac{1}{|Q|}\int_Q w(x)^r\,dx\right)^{1/r}\le C\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)$$

holds for every cube Q in $\mathbb{R}^n$.

It is well known that if $w\in A_p$ with $1<p<\infty$, then $w\in A_r$ for all $r>p$, and $w\in A_q$ for some $1<q<p$. If $w\in A_p$ with $1\le p<\infty$, then there exists $r>1$ such that $w\in RH_r$. It follows from Hölder's inequality that $w\in RH_r$ implies $w\in RH_s$ for all $1<s<r$. Moreover, if $w\in RH_r$, $r>1$, then we have $w\in RH_{r+\varepsilon}$ for some $\varepsilon>0$. We thus write $r_w\equiv\sup\{r>1: w\in RH_r\}$ to denote the critical index of w for the reverse Hölder condition.

We state the following results that we will use frequently in the sequel.

Lemma A ([3])**.** *Let $w\in A_p$, $p\ge 1$. Then, for any cube Q, there exists an absolute constant C such that*

$$w(2Q)\le Cw(Q).$$

In general, for any $\lambda>1$, we have

$$w(\lambda Q)\le C\lambda^{np}w(Q),$$

where C does not depend on Q nor on λ.

Lemma B ([3,4]). *Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants C_1, $C_2 > 0$ such that*

$$C_1 \left(\frac{|E|}{|Q|} \right)^p \leq \frac{w(E)}{w(Q)} \leq C_2 \left(\frac{|E|}{|Q|} \right)^{(r-1)/r}$$

for any measurable subset E of a cube Q.

Lemma C ([5]). *Let $s > 1$, $1 \leq p < \infty$ and $A_p^s = \{w : w^s \in A_p\}$. Then*

$$A_p^s = A_{1+(p-1)/s} \cap RH_s.$$

In particular,

$$A_1^s = A_1 \cap RH_s.$$

Next we shall introduce the Hardy-Littlewood maximal operator and several variants, the fractional integral operator and some function spaces.

Definition 4. *The Hardy-Littlewood maximal operator M is defined by*

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

For $0 < \beta < n$, $r \geq 1$, we define the fractional maximal operator $M_{\beta,r}$ by

$$M_{\beta,r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r\, dy \right)^{1/r}.$$

Let w be a weight. The weighted maximal operator M_w is defined by

$$M_w(f)(x) = \sup_{x \in Q} \frac{1}{w(Q)} \int_Q |f(y)| w(y)\, dy.$$

For $0 < \beta < n$ and $r \geq 1$, we define the fractional weighted maximal operator $M_{\beta,r,w}$ by

$$M_{\beta,r,w}(f)(x) = \sup_{x \in Q} \left(\frac{1}{w(Q)^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r w(y)\, dy \right)^{1/r},$$

where the supremum is taken over all cubes Q containing x.

Definition 5 ([13]). *For $0 < \alpha < n$, the fractional integral operator I_α is defined by*

$$I_\alpha(f)(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Let $1 \le p < \infty$ and w be a weight function. A locally integrable function b is said to be in $BMO_p(w)$ if

$$\|b\|_{BMO_p(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x)^{1-p}\, dx \right)^{1/p} \le C < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(y)\, dy$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. We denote simply by $BMO(w)$ when $p = 1$.

Let $0 < \beta < 1$ and $1 \le p < \infty$. A locally integrable function b is said to be in $Lip_\beta^p(\mathbb{R}^n)$ if

$$\|b\|_{Lip_\beta^p} = \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p\, dx \right)^{1/p} < \infty.$$

We denote simply by $Lip_\beta(\mathbb{R}^n)$ when $p = 1$.

Let $0 < \beta < 1$, $1 \le p < \infty$ and w be a weight function. A locally integrable function b is said to belong to $Lip_\beta^p(w)$ if

$$\|b\|_{Lip_\beta^p(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x)^{1-p}\, dx \right)^{1/p} < \infty.$$

We also denote simply by $Lip_\beta(w)$ when $p = 1$.

Lemma D ([2,10])**.** *(i) Let $w \in A_1$. Then for any $1 \le p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{BMO_p(w)} \le C\|b\|_{BMO(w)}$.*
(ii) Let $0 < \beta < 1$. Then for any $1 \le p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{Lip_\beta^p} \le C\|b\|_{Lip_\beta}$.
(iii) Let $0 < \beta < 1$ and $w \in A_1$. Then for any $1 \le p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{Lip_\beta^p(w)} \le C\|b\|_{Lip_\beta(w)}$.

We are going to conclude this section by defining the weighted Morrey space. For further details, we refer the readers to [6].

Definition 6. *Let $1 \le p < \infty$, $0 < \kappa < 1$ and w be a weight function. Then the weighted Morrey space is defined by*

$$L^{p,\kappa}(w) = \{ f \in L^p_{loc}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty \},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_Q \left(\frac{1}{w(Q)^\kappa} \int_Q |f(x)|^p w(x)\, dx \right)^{1/p}$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Remark. *Equivalently, we could define the weighted Morrey space with balls instead of cubes. Hence we shall use these two definitions of weighted Morrey space appropriate to calculations.*

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights.

Definition 7. *Let $1 \le p < \infty$ and $0 < \kappa < 1$. Then for two weights u and v, the weighted Morrey space is defined by*

$$L^{p,\kappa}(u,v) = \{f \in L^p_{loc}(u) : \|f\|_{L^{p,\kappa}(u,v)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_Q \left(\frac{1}{v(Q)^\kappa}\int_Q |f(x)|^p u(x)\,dx\right)^{1/p}.$$

We shall need the following estimate given in [6].

Theorem E. *If $0 < \beta < n$, $1 < p < n/\beta$, $1/s = 1/p - \beta/n$, $0 < \kappa < p/s$ and $w \in A_{p,s}$, then $M_{\beta,1}$ is bounded from $L^{p,\kappa}(w^p, w^s)$ to $L^{s,\kappa s/p}(w^s)$.*

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{A}{B} \le C$. Moreover, we will denote the conjugate exponent of $r > 1$ by $r' = r/(r-1)$.

3. Proof of Theorem 1

We shall adopt the same method given in [12]. For $0 < \delta < 1$, we define the δ-sharp maximal operator $M^{\#}_\delta$ as

$$M^{\#}_\delta(f) = M^{\#}(|f|^\delta)^{1/\delta},$$

which is a modification of the sharp maximal operator $M^{\#}$ of Fefferman and Stein [14]. We also set $M_\delta(f) = M(|f|^\delta)^{1/\delta}$. Suppose that $w \in A_\infty$, then for any cube Q, we have the following weighted version of the local good λ inequality(see [14])

$$w\big(\{x \in Q : M_\delta f(x) > \lambda, M^{\#}_\delta f(x) \le \lambda\varepsilon\}\big) \le C\varepsilon \cdot w\big(\{x \in Q : M_\delta f(x) > \frac{\lambda}{2}\}\big),$$

for all $\lambda, \varepsilon > 0$. As a consequence, by using the standard arguments(see [14,15]), we can establish the following estimate, which will play an important role in the proof of our main results.

Proposition 3.1. *Let $0 < \delta < 1$, $1 < p < \infty$ and $0 < \kappa < 1$. If $u, v \in A_\infty$, then we have*

$$\|M_\delta(f)\|_{L^{p,\kappa}(u,v)} \le C\|M^{\#}_\delta(f)\|_{L^{p,\kappa}(u,v)}$$

for all functions f such that the left hand side is finite. In particular, when $u = v = w$ and $w \in A_\infty$, then we have

$$\|M_\delta(f)\|_{L^{p,\kappa}(w)} \le C\|M^{\#}_\delta(f)\|_{L^{p,\kappa}(w)}$$

for all functions f such that the left hand side is finite.

Next we are going to prove a series of lemmas which will be used in the proof of our main theorems.

Lemma 3.2. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $w \in A_\infty$. Then for every $0 < \kappa < p/q$, we have*

$$\|M_{\alpha,1,w}(f)\|_{L^{q,\kappa q/p}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Proof. Fix a cube $Q \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{_{2Q}}$, $\chi_{_{2Q}}$ denotes the characteristic function of $2Q$. Since $M_{\alpha,1,w}$ is a sublinear operator, then we have

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}}\Big(\int_Q M_{\alpha,1,w}f(x)^q w(x)\,dx\Big)^{1/q}\\
\le&\frac{1}{w(Q)^{\kappa/p}}\Big(\int_Q M_{\alpha,1,w}f_1(x)^q w(x)\,dx\Big)^{1/q}\\
&+\frac{1}{w(Q)^{\kappa/p}}\Big(\int_Q M_{\alpha,1,w}f_2(x)^q w(x)\,dx\Big)^{1/q}\\
=&I_1+I_2.
\end{aligned}$$

As we know, the fractional weighted maximal operator $M_{\alpha,1,w}$ is bounded from $L^p(w)$ to $L^q(w)$ provided that $w \in A_\infty$. This together with Lemma A yield

$$\begin{aligned}
I_1 &\le C\frac{1}{w(Q)^{\kappa/p}}\Big(\int_{2Q}|f(x)|^p w(x)\,dx\Big)^{1/p}\\
&\le C\|f\|_{L^{p,\kappa}(w)}\cdot\frac{w(2Q)^{\kappa/p}}{w(Q)^{\kappa/p}} \qquad (1)\\
&\le C\|f\|_{L^{p,\kappa}(w)}.
\end{aligned}$$

We now turn to estimate the term I_2. A simple geometric observation shows that for any $x \in Q$, we have

$$M_{\alpha,1,w}(f_2)(x) \le \sup_{R:\, Q\subseteq 3R} \frac{1}{w(R)^{1-\alpha/n}} \int_R |f(y)|w(y)\,dy.$$

When $Q \subseteq 3R$, then by Lemma A, we have $w(Q) \le Cw(R)$. It follows from Hölder's inequality that

$$\begin{aligned}
&\frac{1}{w(R)^{1-\alpha/n}} \int_R |f(y)|w(y)\,dy \\
\le&\frac{1}{w(R)^{1-\alpha/n}} \Big(\int_R |f(y)|^p w(y)\,dy\Big)^{1/p} \Big(\int_R w(y)\,dy\Big)^{1/p'} \\
\le& C\|f\|_{L^{p,\kappa}(w)} \cdot w(R)^{(\kappa-1)/p+\alpha/n} \\
\le& C\|f\|_{L^{p,\kappa}(w)} \cdot w(Q)^{(\kappa-1)/p+\alpha/n},
\end{aligned}$$

where in the last inequality we have used the fact that $(\kappa-1)/p+\alpha/n<0$. Hence

$$I_2 \le C\|f\|_{L^{p,\kappa}(w)} \cdot w(Q)^{(\kappa-1)/p+\alpha/n} w(Q)^{1/q} w(Q)^{-\kappa/p} \le C\|f\|_{L^{p,\kappa}(w)}. \quad (2)$$

Combining the above inequality (2) with (1) and taking the supremum over all cubes $Q \subseteq \mathbb{R}^n$, we obtain the desired result. □

Lemma 3.3. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q = 1/p-\alpha/n$, $0<\kappa<p/q$ and $w\in A_\infty$. Then for any $1<r<p$, we have*

$$\|M_{\alpha,r,w}(f)\|_{L^{q,\kappa q/p}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Proof. With the notations mentioned earlier, we know that

$$M_{\alpha,r,w}(f) = M_{\alpha r,1,w}(|f|^r)^{1/r}.$$

From the definition, we readily see that

$$\|M_{\alpha,r,w}(f)\|_{L^{q,\kappa q/p}(w)} = \|M_{\alpha r,1,w}(|f|^r)\|^{1/r}_{L^{q/r,\kappa q/p}(w)}.$$

Since $1/q = 1/p-\alpha/n$, then for any $1<r<p$, we have $r/q = r/p-\alpha r/n$. Hence, by Lemma 3.2, we can obtain

$$\big\|M_{\alpha r,1,w}(|f|^r)\big\|^{1/r}_{L^{q/r,\kappa q/p}(w)} \le C\big\||f|^r\big\|^{1/r}_{L^{p/r,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

We are done. □

Lemma 3.4. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $w^{q/p}\in A_1$. Then if $0<\kappa<p/q$ and $r_w>\frac{1-\kappa}{p/q-\kappa}$, we have*

$$\|M_{\alpha,1}(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)}\le C\|f\|_{L^{p,\kappa}(w)}.$$

Proof. Fix a ball $B=B(x_0,r_B)\subseteq\mathbb{R}^n$, where $B(x_0,r_B)$ denotes the ball with the center x_0 and radius r_B. We decompose $f=f_1+f_2$, where $f_1=f\chi_{2B}$. Since $M_{\alpha,1}$ is a sublinear operator, then we have

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B M_{\alpha,1}f(x)^q w(x)^{q/p}\,dx\Big)^{1/q}\\
\le&\frac{1}{w(B)^{\kappa/p}}\Big(\int_B M_{\alpha,1}f_1(x)^q w(x)^{q/p}\,dx\Big)^{1/q}\\
&+\frac{1}{w(B)^{\kappa/p}}\Big(\int_B M_{\alpha,1}f_2(x)^q w(x)^{q/p}\,dx\Big)^{1/q}\\
=&I_3+I_4.
\end{aligned}$$

For any function f it is easy to see that

$$M_{\alpha,1}(f)(x)\le CI_\alpha(|f|)(x). \tag{3}$$

From the definition, we can easily check that

$$w\in A_{p,q}\quad\text{if and only if}\quad w^q\in A_{1+q/p'}. \tag{4}$$

Since $w^{q/p}\in A_1$, then by (4), we have $w^{1/p}\in A_{p,q}$. It is well known that the fractional integral operator I_α is bounded from $L^p(w^p)$ to $L^q(w^q)$ whenever $w\in A_{p,q}$(see [9]). This together with Lemma A imply

$$\begin{aligned}
I_3&\le C\frac{1}{w(B)^{\kappa/p}}\Big(\int_{2B}|f(x)|^p w(x)\,dx\Big)^{1/p}\\
&\le C\|f\|_{L^{p,\kappa}(w)}\cdot\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}}\\
&\le C\|f\|_{L^{p,\kappa}(w)}.
\end{aligned}\tag{5}$$

We now turn to deal with I_4. Note that when $x\in B$, $y\in(2B)^c$, then we have $|y-x|\sim|y-x_0|$. Since $q/p>1$ and $w^{q/p}\in A_1$, then by Lemma C, we get $w\in A_1\cap RH_{q/p}$. It follows from the inequality (3), Hölder's inequality and the condition A_p that

$$\begin{aligned}
M_{\alpha,1}(f_2)(x)&\le C\int_{(2B)^c}\frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy\\
&\le C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|f(y)|\,dy
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\cdot|2^{j+1}B|w(2^{j+1}B)^{-1/p}\\
&\quad\cdot\Big(\int_{2^{j+1}B}|f(y)|^p w(y)\,dy\Big)^{1/p}\\
&\le C\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}|2^{j+1}B|^{\alpha/n}w(2^{j+1}B)^{(\kappa-1)/p}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_4 &\le C\|f\|_{L^{p,\kappa}(w)}\cdot\frac{w^{q/p}(B)^{1/q}}{w(B)^{\kappa/p}}\sum_{j=1}^{\infty}|2^{j+1}B|^{\alpha/n}w(2^{j+1}B)^{(\kappa-1)/p}\\
&\le C\|f\|_{L^{p,\kappa}(w)}\cdot\frac{|B|^{-\alpha/n}w(B)^{1/p}}{w(B)^{\kappa/p}}\sum_{j=1}^{\infty}|2^{j+1}B|^{\alpha/n}w(2^{j+1}B)^{(\kappa-1)/p}\\
&= C\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}\frac{|2^{j+1}B|^{\alpha/n}}{|B|^{\alpha/n}}\cdot\frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}}.
\end{aligned}$$

Since $r_w > \frac{1-\kappa}{p/q-\kappa}$, then we can find a suitable number r such that $r > \frac{1-\kappa}{p/q-\kappa}$ and $w \in RH_r$. Consequently, by Lemma B, we can get

$$\frac{w(B)}{w(2^{j+1}B)} \le C\left(\frac{|B|}{|2^{j+1}B|}\right)^{(r-1)/r}.$$

Therefore

$$\begin{aligned}
I_4 &\le C\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}(2^{jn})^{\alpha/n-(r-1)(1-\kappa)/pr}\\
&\le C\|f\|_{L^{p,\kappa}(w)},
\end{aligned}\tag{6}$$

where the last series is convergent since $\alpha/n-(r-1)(1-\kappa)/pr<0$. Combining the above inequality (6) with (5) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we get the desired result. □

It should be pointed out that from the above proof of Lemma 3.4, the same conclusion also holds for the fractional integral operator I_α; that is,

$$\|I_\alpha(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{p,\kappa}(w)}.$$

Lemma 3.5. *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $w^{q/p}\in A_1$. Then if $0<\kappa<p/q$ and $r_w > \frac{1-\kappa}{p/q-\kappa}$, we have*

$$\|M_w(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}.$$

Proof. Fix a cube $Q \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$. Then we have

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}}\Big(\int_Q M_w f(x)^q w(x)^{q/p}\,dx\Big)^{1/q}\\
\le&\frac{1}{w(Q)^{\kappa/p}}\Big(\int_Q M_w f_1(x)^q w(x)^{q/p}\,dx\Big)^{1/q}\\
&+\frac{1}{w(Q)^{\kappa/p}}\Big(\int_Q M_w f_2(x)^q w(x)^{q/p}\,dx\Big)^{1/q}\\
=&I_5+I_6.
\end{aligned}$$

The L^q_w boundedness of M_w and Lemma A imply

$$\begin{aligned}
I_5 &\le C\frac{1}{w(Q)^{\kappa/p}}\Big(\int_{2Q}|f(x)|^q w(x)^{q/p}\,dx\Big)^{1/q}\\
&\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}\cdot\frac{w(2Q)^{\kappa/p}}{w(Q)^{\kappa/p}} \qquad (7)\\
&\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}.
\end{aligned}$$

To estimate I_6, we first note that when $x \in Q$, then by a simple geometric observation, the following inequality holds

$$M_w(f_2)(x) \le \sup_{R:\,Q\subseteq 3R}\frac{1}{w(R)}\int_R |f(y)|w(y)\,dy.$$

Applying Hölder's inequality twice, we can deduce

$$\begin{aligned}
&\frac{1}{w(R)}\int_R |f(y)|w(y)\,dy\\
\le&\frac{1}{w(R)}\Big(\int_R |f(y)|^q w(y)^{q/p}\,dy\Big)^{1/q}\Big(\int_R w(y)^{q'/p'}\,dy\Big)^{1/q'}\\
\le&C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}\cdot|R|^{1/q'-1/p'}w(R)^{(\kappa-1)/p}.
\end{aligned}$$

Since $w^{q/p} \in A_1$, then by Lemma C, we have $w \in A_1 \cap RH_{q/p}$, which yields

$$w^{q/p}(Q)^{1/q} \le C\cdot|Q|^{1/q-1/p}w(Q)^{1/p}.$$

Hence

$$\begin{aligned}
I_6 &\le \frac{w^{q/p}(Q)^{1/q}}{w(Q)^{\kappa/p}}\cdot\sup_{R:\,Q\subseteq 3R}\frac{1}{w(R)}\int_R|f(y)|w(y)\,dy\\
&\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}\cdot\sup_{R:\,Q\subseteq 3R}\frac{|R|^{\alpha/n}}{|Q|^{\alpha/n}}\cdot\frac{w(Q)^{(1-\kappa)/p}}{w(R)^{(1-\kappa)/p}}.
\end{aligned}$$

Since $r_w > \frac{1-\kappa}{p/q-\kappa}$, then we are able to find a positive number r such that $r = \frac{1-\kappa}{p/q-\kappa}$ and $w \in RH_r$. For any cube R with $3R \supseteq Q$, by Lemma B, we thus obtain

$$\frac{w(Q)}{w(3R)} \le C\left(\frac{|Q|}{|3R|}\right)^{(r-1)/r}.$$

Furthermore, from Lemma A, it follows immediately that

$$\frac{w(Q)}{w(R)} \le C\left(\frac{|Q|}{|3R|}\right)^{(r-1)/r}.$$

Therefore

$$\begin{aligned} I_6 &\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \cdot \sup_{R:\, Q\subseteq 3R}\left(\frac{|Q|}{|3R|}\right)^{(1-\kappa)(r-1)/pr-\alpha/n} \\ &\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}. \end{aligned} \tag{8}$$

Combining the above inequality (8) with (7) and taking the supremum over all cubes $Q \subseteq \mathbb{R}^n$, we obtain the desired estimate. □

In order to simplify the notation, we set $M_{0,r,w} = M_{r,w}$. Then we shall prove the following lemma.

Lemma 3.6. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $w^{q/p} \in A_1$ and $r_w > \frac{1-\kappa}{p/q-\kappa}$. Then for every $0 < \kappa < p/q$ and $1 < r < p$, we have*

$$\|M_{r,w}(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}.$$

Proof. Note that

$$M_{r,w}(f) = M_w(|f|^r)^{1/r}.$$

For any $1 < r < p$, we have $r/q = r/p - \alpha r/n$. Since $w^{q/p} \in A_1$, which is equivalent to $w^{\frac{q/r}{p/r}} \in A_1$, by using Lemma 3.5, we thus have

$$\begin{aligned} \|M_{r,w}(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} &= \big\|M_w(|f|^r)\big\|^{1/r}_{L^{q/r,\kappa q/p}(w^{q/p},w)} \\ &\le C\big\||f|^r\big\|^{1/r}_{L^{q/r,\kappa q/p}(w^{q/p},w)} \\ &\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}. \end{aligned}$$

This completes the proof of Lemma 3.6. □

Proposition 3.7. *Let $0 < \delta < 1$, $0 < \alpha < n$, $w \in A_1$ and $b \in BMO(w)$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, we have*

$$M_\delta^{\#}([b,I_\alpha]f)(x) \leq C\|b\|_{BMO(w)}\Big(w(x)M_{r,w}(I_\alpha f)(x)+w(x)^{1-\alpha/n}M_{\alpha,r,w}(f)(x) + w(x)M_{\alpha,1}(f)(x)\Big).$$

Proof. For any given $x\in\mathbb{R}^n$, fix a ball $B=B(x_0,r_B)$ which contains x. We decompose $f=f_1+f_2$, where $f_1=f\chi_{2B}$. Observe that

$$[b,I_\alpha]f(x)=(b(x)-b_{2B})I_\alpha f(x)-I_\alpha((b-b_{2B})f)(x).$$

Since $0<\delta<1$, then for arbitrary constant c, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B\Big||[b,I_\alpha]f(y)|^\delta-|c|^\delta\Big|\,dy\right)^{1/\delta} \qquad (9)\\
\leq&\left(\frac{1}{|B|}\int_B\big|[b,I_\alpha]f(y)-c\big|^\delta\,dy\right)^{1/\delta}\\
\leq& C\left(\frac{1}{|B|}\int_B\big|(b(y)-b_{2B})I_\alpha f(y)\big|^\delta\,dy\right)^{1/\delta}\\
&+C\left(\frac{1}{|B|}\int_B\big|I_\alpha((b-b_{2B})f_1)(y)\big|^\delta\,dy\right)^{1/\delta}\\
&+C\left(\frac{1}{|B|}\int_B\big|I_\alpha((b-b_{2B})f_2)(y)+c\big|^\delta\,dy\right)^{1/\delta}\\
=&\,\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

We are now going to estimate each term separately.
Since $w\in A_1$, then it follows from Hölder's inequality and Lemma D that

$$\begin{aligned}
\mathrm{I}&\leq C\frac{1}{|B|}\int_B\big|(b(y)-b_{2B})I_\alpha f(y)\big|\,dy\\
&\leq C\frac{1}{|B|}\Big(\int_B\big|b(y)-b_{2B}\big|^{r'}w^{1-r'}\,dy\Big)^{1/r'}\Big(\int_B\big|I_\alpha f(y)\big|^r w(y)\,dy\Big)^{1/r}\\
&\leq C\|b\|_{BMO(w)}\frac{w(B)}{|B|}\Big(\frac{1}{w(B)}\int_B\big|I_\alpha f(y)\big|^r w(y)\,dy\Big)^{1/r}\\
&\leq C\|b\|_{BMO(w)}w(x)M_{r,w}(I_\alpha f)(x).
\end{aligned} \qquad (10)$$

Applying Kolmogorov's inequality(see [3, page 485]), Hölder's inequality and Lemma D, we thus have

$$\mathrm{II}\leq C\frac{1}{|B|^{1-\alpha/n}}\int_{2B}\big|(b(y)-b_{2B})f(y)\big|\,dy$$

$$
\begin{aligned}
&\le C\frac{1}{|B|^{1-\alpha/n}}\Big(\int_{2B}\big|b(y)-b_{2B}\big|^{r'}w^{1-r'}\,dy\Big)^{1/r'}\Big(\int_{2B}\big|f(y)\big|^{r}w(y)\,dy\Big)^{1/r}\\
&\le C\|b\|_{BMO(w)}\frac{w(2B)^{1-\alpha/n}}{|2B|^{1-\alpha/n}}\Big(\frac{1}{w(2B)^{1-\alpha r/n}}\int_{2B}\big|f(y)\big|^{r}w(y)\,dy\Big)^{1/r}\\
&\le C\|b\|_{BMO(w)}w(x)^{1-\alpha/n}M_{\alpha,r,w}(f)(x). \qquad (11)
\end{aligned}
$$

It remains to estimate the term III. We first fix the value of c by taking $c=-I_\alpha((b-b_{2B})f_2)(x_0)$, then we obtain

$$
\begin{aligned}
\mathrm{III}&\le C\frac{1}{|B|}\int_B\big|I_\alpha((b-b_{2B})f_2)(y)-I_\alpha((b-b_{2B})f_2)(x_0)\big|\,dy\\
&\le C\frac{1}{|B|}\int_B\int_{(2B)^c}\Big|\frac{1}{|y-z|^{n-\alpha}}-\frac{1}{|x_0-z|^{n-\alpha}}\Big|\big|b(z)-b_{2B}\big|\big|f(z)\big|\,dzdy\\
&\le C\frac{1}{|B|}\int_B\Big(\sum_{j=1}^\infty\int_{2^{j+1}B\backslash 2^jB}\frac{|y-x_0|}{|z-x_0|^{n-\alpha+1}}|b(z)-b_{2B}||f(z)|\,dz\Big)dy\\
&\le C\sum_{j=1}^\infty\frac{r_B}{(2^jr_B)^{n-\alpha+1}}\int_{2^{j+1}B}|b(z)-b_{2B}||f(z)|\,dz\\
&\le C\sum_{j=1}^\infty\frac{1}{2^j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||f(z)|\,dz\\
&\quad+C\sum_{j=1}^\infty\frac{1}{2^j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b_{2^{j+1}B}-b_{2B}||f(z)|\,dz\\
&=\mathrm{IV}+\mathrm{V}.
\end{aligned}
$$

Similarly, by Hölder's inequality and Lemma D, we can get

$$
\begin{aligned}
\mathrm{IV}&\le C\|b\|_{BMO(w)}\sum_{j=1}^\infty\frac{1}{2^j}w(x)^{1-\alpha/n}M_{\alpha,r,w}(f)(x)\\
&\le C\|b\|_{BMO(w)}w(x)^{1-\alpha/n}M_{\alpha,r,w}(f)(x).
\end{aligned}
\qquad (12)
$$

Note that $w\in A_1$, a direct calculation shows that

$$
|b_{2^{j+1}B}-b_{2B}|\le C\|b\|_{BMO(w)}j\cdot w(x). \qquad (13)
$$

Substituting the above inequality (13) into the term V, we thus obtain

$$
\mathrm{V}\le C\|b\|_{BMO(w)}\sum_{j=1}^\infty\frac{j}{2^j}w(x)M_{\alpha,1}(f)(x)\le C\|b\|_{BMO(w)}w(x)M_{\alpha,1}(f)(x). \qquad (14)
$$

Combining the above estimates (10)–(12) with (14) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we get the desired result. $\square$

We are now in a position to give the proof of Theorem 1.

Proof of Theorem 1. For $0 < \alpha < n$ and $1 < p < n/\alpha$, we can choose a positive number r such that $1 < r < p$. Applying Proposition 3.1 and Proposition 3.7, we have

$$\begin{aligned}
&\|[b, I_\alpha]f\|_{L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)}\\
\le\, & C\|M^{\#}_\delta([b, I_\alpha]f)\|_{L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)}\\
\le\, & C\|b\|_{BMO(w)}\Big(\|w(\cdot)M_{r,w}(I_\alpha f)\|_{L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)}\\
&+ \|w(\cdot)^{1-\alpha/n}M_{\alpha,r,w}(f)\|_{L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)}\\
&+ \|w(\cdot)M_{\alpha,1}(f)\|_{L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)}\Big)\\
\le\, & C\|b\|_{BMO(w)}\Big(\|M_{r,w}(I_\alpha f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} + \|M_{\alpha,r,w}(f)\|_{L^{q,\kappa q/p}(w)}\\
&+ \|M_{\alpha,1}(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)}\Big).
\end{aligned}$$

Since $0 < \kappa < p/q$, $w^{q/p} \in A_1$ and $r_w > \frac{1-\kappa}{p/q-\kappa}$, then by using Lemma 3.3, Lemma 3.4 and Lemma 3.6, we thus obtain

$$\begin{aligned}
&\|[b, I_\alpha]f\|_{L^{q,\kappa q/p}(w^{1-(1-\alpha/n)q},w)}\\
\le\, & C\|b\|_{BMO(w)}\Big(\|I_\alpha(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} + \|f\|_{L^{p,\kappa}(w)}\Big)\\
\le\, & C\|b\|_{BMO(w)}\|f\|_{L^{p,\kappa}(w)}.
\end{aligned}$$

Therefore, we complete the proof of Theorem 1. $\square$

4. Proof of Theorem 2

Lemma 4.1. *Let $0 < \alpha+\beta < n$, $1 < p < n/(\alpha+\beta)$, $1/s = 1/p-(\alpha+\beta)/n$ and $w^s \in A_1$. Then for every $0 < \kappa < p/s$ and $1 < r < p$, we have*

$$\|M_{\alpha+\beta,r}(f)\|_{L^{s,\kappa s/p}(w^s)} \le C\|f\|_{L^{p,\kappa}(w^p,w^s)}.$$

Proof. Note that

$$M_{\alpha+\beta,r}(f) = M_{(\alpha+\beta)r,1}(|f|^r)^{1/r}.$$

Since $w^s \in A_1$, then we have $(w^r)^{s/r} \in A_{1+(s/r)/(p/r)'}$, which implies $w^r \in A_{p/r,s/r}$ by (4). Observe that $1/s = 1/p-(\alpha+\beta)/n$, then for any $1 < r < p$,

we have $r/s = r/p - (\alpha+\beta)r/n$. Consequently, by Theorem E, we know that the fractional maximal operator $M_{(\alpha+\beta)r,1}$ is bounded from $L^{p/r,\kappa}(w^p, w^s)$ to $L^{s/r,\kappa s/p}(w^s)$. Therefore

$$\begin{aligned} \|M_{\alpha+\beta,r}(f)\|_{L^{s,\kappa s/p}(w^s)} &= \big\|M_{(\alpha+\beta)r,1}(|f|^r)\big\|^{1/r}_{L^{s/r,\kappa s/p}(w^s)} \\ &\leq C\big\||f|^r\big\|^{1/r}_{L^{p/r,\kappa}(w^p,w^s)} \\ &\leq C\|f\|_{L^{p,\kappa}(w^p,w^s)}. \end{aligned}$$

We are done. □

Lemma 4.2. *Let $0 < \alpha+\beta < n$, $1 < p < n/(\alpha+\beta)$, $1/q = 1/p - \alpha/n$, $1/s = 1/q - \beta/n$ and $w^s \in A_1$. Then for every $0 < \kappa < p/s$, we have*

$$\|M_{\beta,1}(f)\|_{L^{s,\kappa s/p}(w^s)} \leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)}.$$

Proof. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$. Since $M_{\beta,1}$ is a sublinear operator, then we have

$$\begin{aligned} &\frac{1}{w^s(B)^{\kappa/p}}\Big(\int_B M_{\beta,1}f(x)^s w(x)^s\,dx\Big)^{1/s} \\ \leq& \frac{1}{w^s(B)^{\kappa/p}}\Big(\int_B M_{\beta,1}f_1(x)^s w(x)^s\,dx\Big)^{1/s} \\ &+ \frac{1}{w^s(B)^{\kappa/p}}\Big(\int_B M_{\beta,1}f_2(x)^s w(x)^s\,dx\Big)^{1/s} \\ =& J_1 + J_2. \end{aligned}$$

Since $w^s \in A_1$, then by (4), we have $w \in A_{q,s}$. As mentioned in the proof of Lemma 3.4, we know that $M_{\beta,1}$ is bounded from $L^q(w^q)$ to $L^s(w^s)$ whenever $w \in A_{q,s}$. This together with Lemma A give

$$\begin{aligned} J_1 &\leq C\frac{1}{w^s(B)^{\kappa/p}}\Big(\int_{2B} |f(x)|^q w(x)^q\,dx\Big)^{1/q} \\ &\leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)}\frac{w^s(2B)^{\kappa/p}}{w^s(B)^{\kappa/p}} \\ &\leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)}. \end{aligned} \tag{15}$$

To estimate J_2, we note that if $x \in B$ and $y \in (2B)^c$, then $|y - x| \sim |y - x_0|$. It follows from Hölder's inequality and the condition $A_{q,s}$ that

$$M_{\beta,1}(f_2)(x) \leq C\int_{(2B)^c} \frac{|f(y)|}{|x-y|^{n-\beta}}\,dy$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\beta/n}}\int_{2^{j+1}B}|f(y)|\,dy\\
&\leq C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{1-\beta/n}}\Big(\int_{2^{j+1}B}w(y)^{-q'}\,dy\Big)^{1/q'}\\
&\quad\times\Big(\int_{2^{j+1}B}|f(y)|^q w(y)^q\,dy\Big)^{1/q} \qquad (16)\\
&\leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)}\sum_{j=1}^{\infty}w^s(2^{j+1}B)^{\kappa/p-1/s}.
\end{aligned}
$$

Substituting the above inequality (16) into the term J_2, we thus obtain

$$
J_2\leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)}\sum_{j=1}^{\infty}\frac{w^s(B)^{1/s-\kappa/p}}{w^s(2^{j+1}B)^{1/s-\kappa/p}}.
$$

Since $w^s\in A_1$, then we know $w^s\in RH_r$ for some $r>1$. It follows from Lemma B that

$$
\frac{w^s(B)}{w^s(2^{j+1}B)}\leq C\left(\frac{|B|}{|2^{j+1}B|}\right)^{(r-1)/r}.
$$

Therefore

$$
\begin{aligned}
J_2&\leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)}\sum_{j=1}^{\infty}(2^{jn})^{-(1-1/r)(1/s-\kappa/p)}\\
&\leq C\|f\|_{L^{q,\kappa q/p}(w^q,w^s)},
\end{aligned} \qquad (17)
$$

where the last inequality holds since $(1-1/r)(1/s-\kappa/p)>0$. Combining the above estimate (17) with (15), we obtain the desired result. □

Lemma 4.3. *Let $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/q=1/p-\alpha/n$, $1/s=1/q-\beta/n$ and $w^s\in A_1$. Then for every $0<\kappa<p\beta/n$, we have*

$$
\|I_\alpha(f)\|_{L^{q,\kappa q/p}(w^q,w^s)}\leq C\|f\|_{L^{p,\kappa}(w^p,w^s)}.
$$

Proof. Fix a ball $B=B(x_0,r)$ and decompose $f=f_1+f_2$, where $f_1=f\chi_{2B}$. Then we have

$$
\begin{aligned}
&\frac{1}{w^s(B)^{\kappa/p}}\Big(\int_B I_\alpha f(x)^q w(x)^q\,dx\Big)^{1/q}\\
\leq&\frac{1}{w^s(B)^{\kappa/p}}\Big(\int_B I_\alpha f_1(x)^q w(x)^q\,dx\Big)^{1/q}\\
&+\frac{1}{w^s(B)^{\kappa/p}}\Big(\int_B I_\alpha f_2(x)^q w(x)^q\,dx\Big)^{1/q}
\end{aligned}
$$

$$= J_3 + J_4.$$

Since $w^s \in A_1$ and $1 < q < s$, then $w^q \in A_1$, which implies $w \in A_{p,q}$ by (4). The $L^p(w^p)$-$L^q(w^q)$ boundedness of I_α and lemma A yield

$$\begin{aligned} J_3 &\le C\frac{1}{w^s(B)^{\kappa/p}}\Big(\int_{2B}|f(x)|^p w(x)^p\,dx\Big)^{1/p} \\ &\le C\|f\|_{L^{p,\kappa}(w^p,w^s)}\frac{w^s(2B)^{\kappa/p}}{w^s(B)^{\kappa/p}} \\ &\le C\|f\|_{L^{p,\kappa}(w^p,w^s)}. \end{aligned} \tag{18}$$

We now turn to estimate J_4. Since $w \in A_{p,q}$, then similar to the estimate of (16), we can get

$$I_\alpha(f_2)(x) \le C\|f\|_{L^{p,\kappa}(w^p,w^s)} \cdot \sum_{j=1}^{\infty} w^s(2^{j+1}B)^{\kappa/p} w^q(2^{j+1}B)^{-1/q}.$$

As a consequence,

$$J_4 \le C\|f\|_{L^{p,\kappa}(w^p,w^s)} \cdot \sum_{j=1}^{\infty}\frac{w^s(2^{j+1}B)^{\kappa/p}}{w^s(B)^{\kappa/p}} \cdot \frac{w^q(B)^{1/q}}{w^q(2^{j+1}B)^{1/q}}.$$

Since $w^s \in A_1$, then by Lemma B, we thus obtain

$$C \cdot \frac{|B|}{|2^{j+1}B|} \le \frac{w^s(B)}{w^s(2^{j+1}B)}.$$

On the other hand, since $w^s \in A_1$, then by Lemma C, we have $w \in RH_s$. Note that $s > q$, then we can easily verify that $w^q \in RH_{s/q}$. Hence, by using Lemma B again, we get

$$\frac{w^q(B)}{w^q(2^{j+1}B)} \le C\left(\frac{|B|}{|2^{j+1}B|}\right)^{1-q/s}.$$

Therefore

$$\begin{aligned} J_4 &\le C\|f\|_{L^{p,\kappa}(w^p,w^s)}\sum_{j=1}^{\infty}(2^{jn})^{\kappa/p-\beta/n} \\ &\le C\|f\|_{L^{p,\kappa}(w^p,w^s)}, \end{aligned} \tag{19}$$

where the last inequality follows from the fact that $\kappa < p\beta/n$. Combining the above estimate (19) with (18), we conclude the proof of Lemma 4.3. $\square$

Proposition 4.4. *Let $0<\delta<1$, $0<\alpha<n$, $0<\beta<1$, $w\in A_1$ and $b\in Lip_\beta(\mathbb{R}^n)$. Then for all $r>1$ and for all $x\in\mathbb{R}^n$, we have*

$$M_\delta^{\#}([b,I_\alpha]f)(x)\le C\|b\|_{Lip_\beta}\Big(M_{\beta,1}(I_\alpha f)(x)+M_{\alpha+\beta,r}(f)(x)+M_{\alpha+\beta,1}(f)(x)\Big).$$

Proof. As in the proof of Proposition 3.7, we can split the previous expression (9) into three parts and estimate each term respectively. For given $0<\delta<1$, we may choose a sufficiently large number u such that $\delta u>1$ and $0<\delta u'<1$. It follows from Hölder's inequality and Lemma D that

$$\begin{aligned}\mathrm{I}&\le C\Big(\frac{1}{|2B|}\int_{2B}\big|b(y)-b_{2B}\big|^{\delta u}\,dy\Big)^{1/\delta u}\Big(\frac{1}{|B|}\int_B\big|I_\alpha f(y)\big|^{\delta u'}\,dy\Big)^{1/\delta u'}\\&\le C\|b\|_{Lip_\beta}|B|^{\beta/n}\Big(\frac{1}{|B|}\int_B\big|I_\alpha f(y)\big|\,dy\Big)\\&\le C\|b\|_{Lip_\beta}M_{\beta,1}(I_\alpha f)(x).\end{aligned}\tag{20}$$

Using Kolmogorov's inequality, Hölder's inequality and Lemma D, we get

$$\begin{aligned}\mathrm{II}&\le C\frac{1}{|B|^{1-\alpha/n}}\int_{2B}|(b(y)-b_{2B})f(y)|\,dy\\&\le C\|b\|_{Lip_\beta}M_{\alpha+\beta,r}(f)(x).\end{aligned}\tag{21}$$

Following the same lines as that of Proposition 3.7, we thus have

$$\mathrm{III}\le \mathrm{IV}+\mathrm{V},$$

where

$$\mathrm{IV}=C\sum_{j=1}^\infty\frac{1}{2^j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||f(z)|\,dz$$

and

$$\mathrm{V}=C\sum_{j=1}^\infty\frac{1}{2^j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b_{2^{j+1}B}-b_{2B}||f(z)|\,dz.$$

As in the estimate of II, we can also deduce

$$\mathrm{IV}\le C\|b\|_{Lip_\beta}M_{\alpha+\beta,r}(f)(x)\sum_{j=1}^\infty\frac{1}{2^j}\le C\|b\|_{Lip_\beta}M_{\alpha+\beta,r}(f)(x).\tag{22}$$

By Lemma D, it is easy to verify that

$$|b_{2^{j+1}B}-b_{2B}|\le C\|b\|_{Lip_\beta}\cdot j|2^{j+1}B|^{\beta/n}.$$

Hence

$$\begin{aligned}
\mathrm{V} &\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{\infty}\frac{j}{2^j}\cdot\frac{1}{|2^{j+1}B|^{1-(\alpha+\beta)/n}}\int_{2^{j+1}B}|f(z)|\,dz\\
&\leq C\|b\|_{Lip_\beta}M_{\alpha+\beta,1}(f)(x)\sum_{j=1}^{\infty}\frac{j}{2^j}\\
&\leq C\|b\|_{Lip_\beta}M_{\alpha+\beta,1}(f)(x).
\end{aligned}\tag{23}$$

Combining the above estimates (20)–(23) and taking the supremum over all balls $B\subseteq\mathbb{R}^n$, we obtain the desired result. □

Proof of Theorem 2. For $0<\alpha+\beta<n$ and $1<p<n/(\alpha+\beta)$, we can find a number r such that $1<r<p$. Applying Proposition 3.1 and Proposition 4.4, we get

$$\begin{aligned}
\big\|[b,I_\alpha]f\big\|_{L^{s,\kappa s/p}(w^s)} \leq& C\big\|M^{\#}_{\delta}([b,I_\alpha]f)\big\|_{L^{s,\kappa s/p}(w^s)}\\
\leq& C\|b\|_{Lip_\beta}\Big(\big\|M_{\beta,1}(I_\alpha f)\big\|_{L^{s,\kappa s/p}(w^s)}\\
&+\big\|M_{\alpha+\beta,r}(f)\big\|_{L^{s,\kappa s/p}(w^s)}+\big\|M_{\alpha+\beta,1}(f)\big\|_{L^{s,\kappa s/p}(w^s)}\Big).
\end{aligned}$$

Let $1/q=1/p-\alpha/n$ and $1/s=1/q-\beta/n$. Since $w^s\in A_1$, then by (4), we have $w\in A_{p,s}$. Since $0<\kappa<\min\{p/s,p\beta/n\}$, by Theorem E, Lemma 4.1, Lemma 4.2 and Lemma 4.3, we thus obtain

$$\begin{aligned}
\big\|[b,I_\alpha]f\big\|_{L^{s,\kappa s/p}(w^s)} &\leq C\|b\|_{Lip_\beta}\Big(\big\|I_\alpha(f)\big\|_{L^{q,\kappa q/p}(w^q,w^s)}+\|f\|_{L^{p,\kappa}(w^p,w^s)}\Big)\\
&\leq C\|b\|_{Lip_\beta}\|f\|_{L^{p,\kappa}(w^p,w^s)}.
\end{aligned}$$

This completes the proof of Theorem 2. □

5. Proof of Theorem 3

Lemma 5.1. *Let $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/q=1/p-\alpha/n$, $1/s=1/q-\beta/n$ and $w^{s/p}\in A_1$. Then if $0<\kappa<p/s$, $r_w>\frac{1}{p/s-\kappa}$, we have*

$$\big\|M_{\beta,1}(f)\big\|_{L^{s,\kappa s/p}(w^{s/p},w)}\leq C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}.$$

Proof. Fix a ball $B=B(x_0,r_B)\subseteq\mathbb{R}^n$. Let $f=f_1+f_2$, where $f_1=f\chi_{_{2B}}$. Since $M_{\beta,1}$ is a sublinear operator, then we have

$$\frac{1}{w(B)^{\kappa/p}}\Big(\int_B M_{\beta,1}f(x)^s w(x)^{s/p}\,dx\Big)^{1/s}$$

$$
\begin{aligned}
\leq & \frac{1}{w(B)^{\kappa/p}}\Big(\int_B M_{\beta,1}f_1(x)^s w(x)^{s/p}\,dx\Big)^{1/s} \\
& + \frac{1}{w(B)^{\kappa/p}}\Big(\int_B M_{\beta,1}f_2(x)^s w(x)^{s/p}\,dx\Big)^{1/s} \\
= & K_1 + K_2.
\end{aligned}
$$

Since $w^{s/p} \in A_1$, then by (4), we can get $w^{1/p} \in A_{q,s}$. The $L^q(w^q)$-$L^s(w^s)$ boundedness of $M_{\beta,1}$ and Lemma A imply

$$
\begin{aligned}
K_1 &\leq C\frac{1}{w(B)^{\kappa/p}}\Big(\int_{2B}|f(x)|^q w(x)^{q/p}\,dx\Big)^{1/q} \\
&\leq C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}\frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\
&\leq C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}.
\end{aligned}
\tag{24}
$$

We turn to deal with the term K_2. Since $w^{1/p} \in A_{q,s}$, as in the estimate of (16), we thus have

$$
M_{\beta,1}(f_2)(x) \leq C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}\sum_{j=1}^{\infty} w(2^{j+1}B)^{\kappa/p}\cdot w^{s/p}(2^{j+1}B)^{-1/s}.
$$

Hence

$$
K_2 \leq C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}\sum_{j=1}^{\infty}\frac{w(2^{j+1}B)^{\kappa/p}}{w(B)^{\kappa/p}}\cdot\frac{w^{s/p}(B)^{1/s}}{w^{s/p}(2^{j+1}B)^{1/s}}.
$$

Since $w^{s/p} \in A_1$ and $s > p$, then by Lemma C, we have $w \in A_1 \cap RH_{s/p}$. Furthermore, by Lemma B, we can get

$$
C\cdot\frac{|B|}{|2^{j+1}B|} \leq \frac{w(B)}{w(2^{j+1}B)}.
$$

Since $r_w > \frac{1}{p/s-\kappa}$, then we can find a positive number r such that $r > \frac{1}{p/s-\kappa}$ and $w \in RH_r$. Also we note that $s/p > 1$, then it is easy to see that

$$
w^{s/p} \in RH_{rp/s}.
$$

By using Lemma B again, we thus obtain

$$
\frac{w^{s/p}(B)}{w^{s/p}(2^{j+1}B)} \leq C\left(\frac{|B|}{|2^{j+1}B|}\right)^{1-s/(rp)}.
$$

Therefore

$$
\begin{aligned}
K_2 &\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \sum_{j=1}^{\infty} (2^{jn})^{-1/s+\kappa/p+1/(rp)} \\
&\le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)},
\end{aligned} \tag{25}
$$

where in the last inequality we have used the fact that $-1/s+\kappa/p+1/(rp) < 0$. Combining the above inequality (25) with (24) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we complete the proof of Lemma 5.1. $\square$

Proposition 5.2. *Let $0 < \delta < 1$, $0 < \alpha < n$, $0 < \beta < 1$, $w \in A_1$ and $b \in Lip_\beta(w)$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, we have*

$$
\begin{aligned}
M_\delta^{\#}([b, I_\alpha]f)(x) \le & C\|b\|_{Lip_\beta(w)} \Big(w(x)^{1+\beta/n} M_{\beta,1}(I_\alpha f)(x) \\
& + w(x)^{1-\alpha/n} M_{\alpha+\beta,r,w}(f)(x) + w(x)^{1+\beta/n} M_{\alpha+\beta,1}(f)(x) \Big).
\end{aligned}
$$

Proof. Again, as in the proof of Proposition 3.7, we will split the previous expression (9) into three parts and estimate each term respectively. For given $0 < \delta < 1$, we are able to find a real number $1 < u < 2$ such that $0 < \delta u < \delta u' < 1$. It follows from Hölder's inequality and Lemma D that

$$
\begin{aligned}
\mathrm{I} &\le C\Big(\frac{1}{|2B|}\int_{2B} \big|b(y) - b_{2B}\big|^{\delta u}\, dy\Big)^{1/\delta u} \Big(\frac{1}{|B|}\int_B \big|I_\alpha f(y)\big|^{\delta u'}\, dy\Big)^{1/\delta u'} \\
&\le C\Big(\frac{1}{|2B|}\int_{2B} \big|b(y) - b_{2B}\big|\, dy\Big)\Big(\frac{1}{|B|}\int_B \big|I_\alpha f(y)\big|\, dy\Big) \\
&\le C\|b\|_{Lip_\beta(w)} \frac{w(2B)^{1+\beta/n}}{|2B|} \cdot \Big(\frac{1}{|B|}\int_B \big|I_\alpha f(y)\big|\, dy\Big) \\
&\le C\|b\|_{Lip_\beta(w)} w(x)^{1+\beta/n} M_{\beta,1}(I_\alpha f)(x).
\end{aligned} \tag{26}
$$

As before, by Kolmogorov's inequality, Hölder's inequality and Lemma D, we thus obtain

$$
\begin{aligned}
\mathrm{II} &\le C\frac{1}{|B|^{1-\alpha/n}} \int_{2B} |(b(y) - b_{2B})f(y)|\, dy \\
&\le C\|b\|_{Lip_\beta(w)} w(x)^{1-\alpha/n} M_{\alpha+\beta,r,w}(f)(x).
\end{aligned} \tag{27}
$$

Again, by using the same arguments as that of Proposition 3.7, we thus have

$$
\mathrm{III} \le \mathrm{IV}+\mathrm{V},
$$

where

$$\mathrm{IV} = C\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||f(z)|\,dz$$

and

$$\mathrm{V} = C\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b_{2^{j+1}B}-b_{2B}||f(z)|\,dz.$$

Similar to the estimate of II, we can also get

$$\mathrm{IV} \le C\|b\|_{Lip_\beta(w)}w(x)^{1-\alpha/n}M_{\alpha+\beta,r,w}(f)(x). \tag{28}$$

Observe that $w \in A_1$, then by Lemma D, a simple calculation gives that

$$|b_{2^{j+1}B}-b_{2B}| \le C\|b\|_{Lip_\beta(w)}j\cdot w(x)w(2^{j+1}B)^{\beta/n}.$$

Therefore

$$\begin{aligned}
\mathrm{V} &\le C\|b\|_{Lip_\beta(w)}\sum_{j=1}^{\infty}\frac{j}{2^j}\cdot\frac{w(x)w(2^{j+1}B)^{\beta/n}}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|f(z)|\,dz\\
&\le C\|b\|_{Lip_\beta(w)}\sum_{j=1}^{\infty}\frac{j}{2^j}\cdot w(x)^{1+\beta/n}\frac{1}{|2^{j+1}B|^{1-(\alpha+\beta)/n}}\int_{2^{j+1}B}|f(z)|\,dz\\
&\le C\|b\|_{Lip_\beta(w)}w(x)^{1+\beta/n}M_{\alpha+\beta,1}(f)(x)\sum_{j=1}^{\infty}\frac{j}{2^j}\\
&\le C\|b\|_{Lip_\beta(w)}w(x)^{1+\beta/n}M_{\alpha+\beta,1}(f)(x).
\end{aligned}\tag{29}$$

Summarizing the estimates (26)–(29) derived above and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we get the desired result. □

Finally let us give the proof of Theorem 3.

Proof of Theorem 3. As before, for $0 < \alpha+\beta < n$ and $1 < p < n/(\alpha+\beta)$, we are able to choose a number r such that $1 < r < p$. By Proposition 3.1 and Proposition 5.2, we have

$$\begin{aligned}
&\|[b,I_\alpha]f\|_{L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)}\\
\le\,& C\|M_\delta^{\#}([b,I_\alpha]f)\|_{L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)}\\
\le\,& C\|b\|_{Lip_\beta(w)}\Big(\|w(\cdot)^{1+\beta/n}M_{\beta,1}(I_\alpha f)\|_{L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)}\\
&+\|w(\cdot)^{1-\alpha/n}M_{\alpha+\beta,r,w}(f)\|_{L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)}\\
&+\|w(\cdot)^{1+\beta/n}M_{\alpha+\beta,1}(f)\|_{L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)}\Big)
\end{aligned}$$

$$\leq C\|b\|_{Lip_\beta(w)}\Big(\|M_{\beta,1}(I_\alpha f)\|_{L^{s,\kappa s/p}(w^{s/p},w)}+\|M_{\alpha+\beta,r,w}(f)\|_{L^{s,\kappa s/p}(w)}$$
$$+\|M_{\alpha+\beta,1}(f)\|_{L^{s,\kappa s/p}(w^{s/p},w)}\Big).$$

Let $1/q=1/p-\alpha/n$ and $1/s=1/q-\beta/n$. Since $r_w>\frac{1}{p/s-\kappa}>\frac{1-\kappa}{p/s-\kappa}>\frac{1-\kappa}{p/q-\kappa}$, then by using Lemma 3.3, Lemma 3.4 and Lemma 5.1, we thus obtain

$$\begin{aligned}&\big\|[b,I_\alpha]f\big\|_{L^{s,\kappa s/p}(w^{1-(1-\alpha/n)s},w)}\\ \leq{}& C\|b\|_{Lip_\beta(w)}\Big(\|I_\alpha(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)}+\|f\|_{L^{p,\kappa}(w)}\Big)\\ \leq{}& C\|b\|_{Lip_\beta(w)}\|f\|_{L^{p,\kappa}(w)}.\end{aligned}$$

Therefore, we conclude the proof of Theorem 3. □

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